Transitional Flowing of Fluids Simulated With a Simple Graphics Model.

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Abstract

This paper considers the problem of liquid flow simulated in a tube using a simplified graphics model capable of fast rendering. A liquid model is represented by a set of particles whose position, speed and acceleration are derived from Navier-Stockes equations. The equations are solved by explicit numerical integration in discrete time steps. Finally, each particle used in the model is represented by a sphere or a metaball for smooth visualization of liquid surfaces.

1. Introduction

A simple water system has a tree topology structure without cycles while connections between nodes are peace-wise linear. The orientation of small linear segments is considered to be arbitrary in a 3-D. The focus of this paper is on a fluid flow in a linear peace segment. It is an initial problem where the fluid speed at the beginning of a tube is proportional to a pump power. As the fluid particle moves further from the pump its speed decreases or eventually particle can not move any further. Such places are good candidates for the installation of an additional pump.

2. Mathematical Model

Let Ω be a Lipschitz continuous bounded domain of R^n (set of a real n- dimensional vectors) with boundary $\partial\Omega$.

In Cartesian coordinates $O\{x_1, x_2, x_3\}$ the velocity of flowing particle in Ω can be written by the following Navier-Stokes equations:

$$\boxed{\begin{gathered}
\mathbf{u}:\Omega\times\langle 0,T\rangle \to R^n;\; p:\Omega\times\langle 0,T\rangle \to R \\
\frac{\partial \mathbf{u}}{\partial t} - \mu\nabla^2\mathbf{u} + \sum_{i=1}^{n} u_i D_i \mathbf{u} - \nabla p = \mathbf{f} \text{ in } Q := \Omega\times\langle 0,T\rangle, \\
\operatorname{div}\mathbf{u} = 0 \text{ in } Q, \\
\mathbf{u} = 0 \text{ on } \partial\Omega\times(0,\mathrm{T}), \\
\mathbf{u}(x,0) = \mathbf{u}_0(x) \text{ in } \Omega,
\end{gathered}}$$

(2.1)

where the position $\mathbf{u} = (u_1,\dots,u_n)$ is a vector function of a time and is the velocity of the fluid, $\mu > 0$ is its kinematic viscosity (assumed to be a constant), $\nabla^2\mathbf{u} = \frac{\partial^2\mathbf{u}}{\partial x_1^2} + \dots + \frac{\partial^2\mathbf{u}}{\partial x_n^2}$, p is its kinematic pressure and $\mathbf{f} = (f_1,\dots,f_n)$, (n = 3) represents a density of body forces per unit mass (gravity, for instance). We set $p = \frac{P}{\rho}$ (kinematic pressure), where P is a pressure in a fluid and ρ is a density. The simplified model used in our simulation assumed Ω as a tube and constant kinematic pressure on each point of Ω in every time.

We say that the solution of Navier-Stokes problem is transitional flowing of fluid if this solution is changing with any time (it is time dependent, (2.1)).

2.1. Approximation With a Finite Differences

2.1.1. Definitions:

We define $D(\Omega)$ to be a linear space of functions infinitely differentiable and with compact support on Ω. Now, let $D'(\Omega)$ denote the dual space of $D(\Omega)$.

Let $\alpha = (\alpha_1,\dots,\alpha_n) \in N^n$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$. For $\mathbf{u}$ in $D'(\Omega)$, we define $\partial^\alpha \mathbf{u}$ in $D'(\Omega)$ as follows:

$$\partial^\alpha \mathbf{u} = D^\alpha \mathbf{u} = (D_1^{\alpha_1} \dots D_n^{\alpha_n})\mathbf{u} = \frac{\partial^{|\alpha|}\mathbf{u}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

For $m \in N$ (positive integer numbers) $\mathrm{p} \in R$ and $1 \le \mathrm{p} \le \infty$, we define the Sobolev space:

$$W^{m,\mathrm{p}}(\Omega) = \left\{ \mathrm{v} \in \mathrm{L}^{\mathrm{p}}(\Omega);\ \partial^\alpha \in \mathrm{L}^{\mathrm{p}}(\Omega),\ \forall |\alpha| \le m \right\}$$

which is a Banach space for the norm $\|\mathbf{u}\|_{m,\mathrm{p},\Omega} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha \mathbf{u}|^{\mathrm{p}} dx \right)^{1/\mathrm{p}}$. When p=2, $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$, it is the Hilbert space for the scalar product: $(\mathbf{u},\mathbf{v})_{m,\Omega} = \sum_{|\alpha| \le m} \int_\Omega D^\alpha \mathbf{u}(x) D^\alpha \mathbf{v}(x) dx$.

As $D(\Omega) \subset H^m(\Omega)$, we define $H_0^m(\Omega) = \overline{D}(\Omega)^{H^m(\Omega)}$, i.e. $H_0^m(\Omega)$ is the closure of $D(\Omega)$ for the norm $\| \,.\, \|_{m,\,\Omega}$.

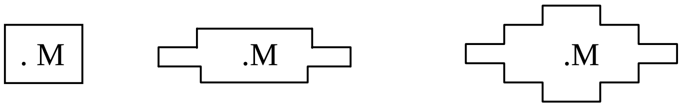
Let $\mathbf{h} = (h_1,\dots,h_n)$ be a vector, where h_i - is step in direction x_i and $0 \le h_i < \infty$.

i) $\mathbf{h}_i = (0,\dots,h_i,\dots,0)$

ii) $\mathrm{R_h}$ is a set of points from R^{n} of the type $j_1\mathbf{h}_1 + \dots + j_n\mathbf{h}_n$, where j_i is from Z (integer numbers).

iii) The set $\sigma_h(\mathbf{M}) = \prod_{i=1}^{n} (\mu_i - h_i/2, \mu_i + h_i/2)$ (Cartesians product of the intervals) is called a block. Here we take $\mathbf{M} = (\mu_1,\dots,\mu_n)$.

iv) $\sigma_h(\mathbf{M},r)$ - a class of type $\bigcup\limits_{\substack{1\le i\le n\\ -r\le\alpha\le r}}\sigma_{\mathrm{h}}(\mathbf{M}+(\alpha/2)\mathbf{h_i})$ (see following picture)



$r=0$ $r=1$ $r=2$

v) $\omega_{h\mathbf{M}}(x)$ is a characteristic function of $\sigma_h(\mathbf{M})$

vi) Denote δ_{ih} (or δ_i) a finite-difference operator by

$$(\delta_i\varphi)(x)=\frac{\varphi(x+\mathbf{h}_i/2)-\varphi(x-\mathbf{h}_i/2)}{h_i}.$$

If $j=(j_1,\dots,j_n)\in N^n$ is a multiindex then δ_h^j (or simple δ^j) is defined $\delta^j=\delta_1^{j_1}\dots\delta_n^{j_n}$.

vii) For each opened set $\Omega\subset R^n$ and positive integer r we define the following sets:

$$\mathring{\Omega}_h^{r}=\left\{\mathbf{M}\in \mathrm{R_h},\sigma_h(\mathbf{M},r)\subset\Omega\right\}$$

$$\Omega_h^r=\left\{\mathbf{M}\in \mathrm{R_h},\sigma_h(\mathbf{M},r)\cap\Omega\neq\varnothing\right\}$$

viii) Further define finite-difference operators ∇_{ih} , $\overline{\nabla}_{ih}$ (simpler ∇_i , $\overline{\nabla}_i$)

$$\nabla_{ih}\varphi(x)=\frac{\varphi(x+h_i)-\varphi(x)}{h_i}$$

$$\overline{\nabla}_{ih}\varphi(x)=\frac{\varphi(x)-\varphi(x-h_i)}{h_i}$$

2.1.2. Space W_h :

The symbol W_h denotes the staircase functions space with the elements:

$$\mathbf{u}_h(x)=\sum_{\mathbf{M}\in\mathring{\Omega}_h^1}\mathbf{u}_h(\mathbf{M})\omega_{h\mathbf{M}}(x);\ \mathbf{u}_h(\mathbf{M})\in R^n$$

The functions $\omega_{h\mathbf{M}}$ for $\mathbf{M}\in\mathring{\Omega}_h^1$ are linearly independent, they create a base and generate the space W_h. The dimension of this space is equal to $n.N(h)$ and it is finite. Where $N(h)$ is number of points $\mathbf{M}\in\mathring{\Omega}_h^1$. This facts give us that the numbers of base elements is equal to dimension of W_h

2.1.3. Scalar products:

It is important to define the following scalar products for the variation interpretation of our problem:

i) $(\mathbf{u},\mathbf{v})=\int\limits_{\Omega}\mathbf{u}(x)\mathbf{v}(x)dx$

ii) $((\mathbf{u},\mathbf{v}))=\sum\limits_{i=1}^{n}(D_i\mathbf{u},D_i\mathbf{v})$

iii) $((\mathbf{u}_h,\mathbf{v}_h))_h = \sum_{i=1}^{n}(D_{ih}\mathbf{u}_h, D_{ih}\mathbf{v}_h)$;

$D_{ih}\mathbf{u}_h(x) = D_i\mathbf{u}_h(x)$; for triangulation.

We have the following relations:

i) $(\delta_{ih}\mathbf{u}_h, \delta_{ih}\mathbf{w}_h) \to (D_i\mathbf{v}, D_i\mathbf{w})$

ii) $((\mathbf{u}_h,\mathbf{w}_h))_h \to ((\mathbf{v},\mathbf{w}))$

iii) $(\mathbf{f},\mathbf{w}_h) \to (\mathbf{f},\mathbf{w})$

Due to these relations we can approximate Navier-Stokes problem.

2.1.4. Form b_h:

The transitional in the Navier-Stokes equations will be expressed by the form $b_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = b_h^{'}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) + b_h^{''}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)$, where

$b_h^{'}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = 2^{-1}\sum_{i,j=1}^{n}\int_{\Omega} u_{ih}(D_{ih}v_{jh})w_{jh}dx$ and $b_h^{''}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = -b_h^{'}(\mathbf{u}_h,\mathbf{w}_h,\mathbf{v}_h)$.

3. Numerical Solution

The existence and uniqueness of the solution of problem (2.1) can be found in [6] or [1]. Idea of the numerical solution of (2.1) is based on the division of domain Ω and the time interval <0,T>. The division of the domain Ω can be represented with the space $\overset{o}{\Omega}{}^{1}_{h}$.

3.1. The Time Axis Division

Let us choose a positive integer N. Let $k = \frac{T}{N}$ be the corresponding time-step, and $t_m = k.m$ for $m = 1,\dots,N$. The solution $\mathbf{u}_h : \langle 0,T\rangle \to W_h$ and $(m-1)k \le t \le mk$ we have $\mathbf{u}_h(t) =: \mathbf{u}_h^m = \mathbf{u}_h(t_m)$ for $m = 1,\dots,N$ and $\mathbf{f}^m = \frac{1}{k}\int_{(m-1)k}^{mk}\mathbf{f}(t)dt$; $\mathbf{f}^m \in L^2(\Omega)$. After this time axis division we can rewrite the Navier-Stokes problem into the discrete form:

$$\boxed{\frac{1}{k}(\mathbf{u}_h^m - \mathbf{u}_h^{m-1}, \mathbf{v}_h) + \mu\,((\mathbf{u}_h^m,\mathbf{v}_h))_h + b_h(\mathbf{u}_h^{m-1},\mathbf{u}_h^m,\mathbf{v}_h) - (\pi_h^m, D_h\mathbf{v}_h) = (\mathbf{f}^m,\mathbf{v}_h) \quad \forall \mathbf{v}_h \in W_h}$$

(3.1) The variation interpretation of Navier Stokes problem

In the equation (3.1) the π_h^m is a pressure at the time $(m-1)k \le t \le mk$ expressed by

$$\boxed{\pi_h = \sum_{\mathbf{M}\in\overset{o}{\Omega}{}^{1}_{h}} \vec{\eta}_{\mathbf{M}}\,\omega_{h\mathbf{M}};\ \vec{\eta}_{\mathbf{M}} \in R.}$$

(3.2)

Next, $D_h\mathbf{v}_h = \sum_{\mathbf{M}\in \overset{o}{\Omega}{}_h^1} (D_h\mathbf{v}_h(\mathbf{M}))\omega_{h\mathbf{M}}$, where $D_h\mathbf{v}_h(\mathbf{M}) = \sum_{i=1}^{n} \nabla_{ih} u_{ih}(\mathbf{M});\ \forall \mathbf{M} \in \overset{\mathbf{o}}{\Omega}{}_h^1$.

The equation (3.1) has N solutions for each $m = 1,\dots,N$. The solution $\mathbf{u}_h^m$ approximates the solution $\mathbf{u}(t)$ of the problem ((2.1) see [6]). on the interval $(m-1)k \le t \le mk$.

3.2. Algorithm Arrow-Hurwicz

We construct two sequences $\mathbf{u}_h^{m,r} \in W_h$; $\pi_h^{m,r} \in X_h$ (X_h - function space with the elements (3.2). For the base of this space look at the chapter 4) to solve equation (3.1).
The algorithm starts by arbitrary elements $\mathbf{u}_h^{m,0} \in W_h$; $\pi_h^{m,0} \in X_h$.
If we have $\mathbf{u}_h^{m,r}$; $\pi_h^{m,r}$ then we calculate $\mathbf{u}_h^{m,r+1} \in W_h$; $\pi_h^{m,r+1} \in X_h$ from the equations:

$$\boxed{\begin{gathered} k((\mathbf{u}_h^{m,r+1} - \mathbf{u}_h^{m,r}, \mathbf{v}_h))_h + (\mathbf{u}_h^{m,r}, \mathbf{v}_h) + k\mu((\mathbf{u}_h^{m,r}, \mathbf{v}_h))_h + \\ + kb_h(\mathbf{u}_h^{m-1}, \mathbf{u}_h^{m,r}, \mathbf{v}_h) - k(\pi_h^{m,r}, D_h\mathbf{v}_h) = (\mathbf{u}_h^{m-1}, \mathbf{v}_h) + k(\mathbf{f}^m, \mathbf{v}_h)\ \forall \mathbf{v}_h \in W_h \\ \alpha(\pi_h^{m,r+1} - \pi_h^{m,r}, \mathbf{q}_h) + \rho(D_h\mathbf{u}_h^{m,r+1}, \mathbf{q}_h) = 0\ \forall \mathbf{q}_h \in X_h \end{gathered}}$$

(3.3)

Where ρ, α are constants satisfying the inequalities:

$$\boxed{0 < \rho < 2\alpha\mu / (\alpha\mu^2 + n)}$$

Inequality (3.4)

for n =2 or 3 (the dimension of the space R^n).
Theorem 3.1: Let ρ, α be constants satisfying (3.4) then :

$$\mathbf{u}_h^{m,r} \xrightarrow{r\to\infty} \mathbf{u}_h^m \text{ in } W_h$$

$$\pi_h^{m,r} \xrightarrow{r\to\infty} \pi_h^m \text{ in } X_h / R \text{ (it is the factor space)}$$

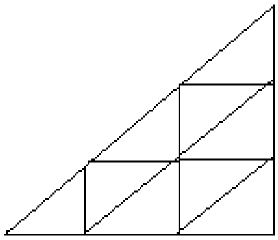
The existence and uniqueness of solution (3.3): $\mathbf{u}_h^{m,r+1} \in W_h$ solving (3.3) is in [6]. □

4. Triangulation

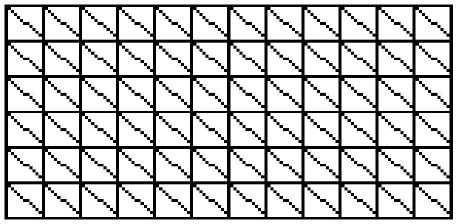
The domain Ω can be very complicated and one needs to approximate Ω. The best way for approximation Ω is a triangulation of the domain Ω. Let be Ω subset R^2. Assume that we have an inaccurate triangulation $\aleph_1$ of Ω. Then we applicate the following algorithm 4.1.

4.1. Algorithm

i) Each side of a triangle from $\aleph_1$ is divided to k the same parts. So we get (k -1) points on the each side of the triangle and k^2 new triangles.
ii) By the correct connection (see pic.4.1.) of points we get the finer triangulation of Ω.



(pic. 4.1.)
$k = 3$



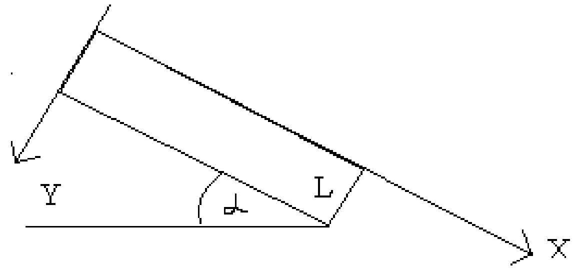
(pic. 4.2.)
Triangulation of Ω, where Ω is a rectangle

We want to know a basis of the spaces W_h , X_h to solve equation (3.3). The centers of all sides of the triangles we call *knops*. One triangle has three knops $\mathbf{P}_{j,k} = (P^j_{k,1}, P^j_{k,2})$ for k=1,2,3 ; (j mean the number of a triangle). Denote all knops from triangulation by the numbers $i = 1,\ldots, N_m$ and $\mathbf{P}^i_{j,k} = (P^{j,i}_{k,1}, P^{j,i}_{k,2})$.
Now we define function ω_i. It represents a plane defined by the three points $(\mathbf{P}_{j,k},1)$; $(\mathbf{P}_{j,l},0)$ for $l = 1,2,3$ and $k \neq l$ (where $\mathbf{P}_{j,k} = \mathbf{P}^i_{j,k}$) for all triangle from the triangulation including knop i . For other triangles ω_i is equal to zero. It is easy to check that ω_i creates the base of X_h for $i = 1,\ldots, N_m$. If n=2, then the set $\{(\omega_i,0),(0,\omega_i);\ i = 1,\ldots, N_m\}$ creates the base of space W_h.

5. Animation Flow

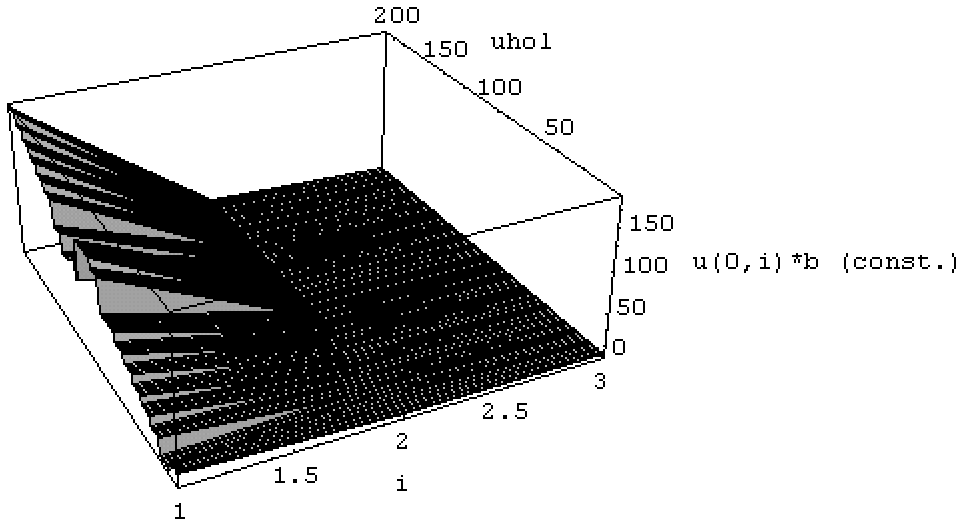
Now, we take domain Ω as a rectangle and we divide it, approximately, to $1\,mm^2$ rectangle parts. The triangulation includes triangles with 1mm, 1mm, $\sqrt{2}$ long sides, approximately. The solution of Navier-Stokes problem found by the previous method is defined on these triangles. We denote the speed $\mathbf{u}_h$ (the solution of (2.1)) in the centroid of triangle by A. The centroid representate all points in the triangle. Let define metacircle for every triangle, with center in centroid of this triangle. The animation shows a fluid flow in a pipe and also its outlet. By the other words: The animation shows a that metacircle (metaballs) flow with the calculated speed vector. For this animation it is need to specify the function $\mathbf{f}$:

Let have a pipe parallel to axis X. (see the picture (5.1)). In the picture the high of pipe is L, and α is a lead angle from earth. The gravitation and lead angle is acting to each molecule from fluid. From that we have $\mathbf{f} = (g\sin\alpha, g\cos\alpha)$.



(pic. 5.1.)

In the first case we take the viscosity $\mu = 0.01$, pressure p=$10^5 \mathbf{N} m^{-2}$ in every triangle, which describe water. If we would like to animate the flow of a mercury then we set: $\mu = 0.03$ and p=$10^5 \mathbf{N} m^{-2}$.
One result:



(5.2) The velocity of fluid at point (i) of pipe with a time 0 and with continuously changed angle (in an one dimension, n=1).

The mass of the metacircles in a pipe is displayed by method described in [4] and each metacircle has evaluated velocity from Navier Stokes equations.

6. Conclusion

This paper shows one method for the generating of a nearly realistic animation of the fluid flow in Lipschitz continuos domain. To solve (2.1) it is necessary put all base elements (see the part: ‘Triangulation“) of space W_h instead of the vector $\mathbf{v}_h$ and all bases elements of space X_h instead of $\mathbf{q}_h$ into (3.3). Then we get N_m equations with N_m unknowns. This method can be applied in the Virtual Reality, water-works or traffic engineering. Why in traffic engineering? In the concretized model from the part 5 we can change the domain Ω so that it will approximate some street from city. So such changed model can show for example the rain water flowing away from the street. We can see this situation in a computer simulation, before building the street. The simulation of water flow in a mountain should prevent people before a flood. But, if we want very realistic simulation we must solve a large system of equations. It needs many time to calculate the result.

7. References

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